

Appendix A

Review of Additional Papers

Since this thesis outdates the survey presented in chapter 3 by more than two years we felt the need to update the survey. We therefore reviewed 70 new papers covering the period from September 2000 up until December 2002. The information gathered from these papers has not been digested as thoroughly as is the case for the previously reviewed papers. Instead we have focused on providing an overview of recent work by presenting the reviewed papers in tables similar to tables 3.1 and 3.5.

Recall from chapter 3 that all computer vision-based human MoCap papers can be described with respect to the general structure in figure 3.1. The structure contains four processes; initialisation, tracking, pose estimation, and recognition. All the papers reviewed for this appendix are listed in table A.1 with respect to the structure. The papers are ordered first by the year of publication and second by the surname of the first author. Four columns allow the clarification of the contributions of the papers within the four processes. The location of the reference number (in brackets) indicates the main topic of the work and an asterisk (*) indicates that the paper also describes work at an interesting level regarding this process.

Besides listing the papers with respect to the four processes we furthermore list the subset of papers that apply a 'direct model use' in the pose estimation process, see section 3.5.3. This means that an *a priori* model of the human is used as the model representing the observed subject. We list this subset primarily due to the fact that we also apply a direct model use in this thesis. In table A.2 the subset is listed. The table consists of 7 columns. The first and second column states the publication year and first author. The third shows how the human is modelled. The fourth shows the number of parts in the model. The fifth shows which part of the subject is analysed. The sixth shows which abstraction level the system works at and the last shows the dimensionality of the pose estimated data. More details can be found in chapter 3.

As stated above we have not digested the papers in this appendix as thoroughly

as previously and will therefore not discuss the newly reviewed papers. However, we will mention one important issue that became clear during the review of the recent papers. This is the trend towards applying sequential Monte Carlo (SMC) methods in computer vision-based MoCap in general and in pose estimation in particular. The motivation is primarily to handle the often incorrect assumption that the estimated state of an object and its uncertainties can be modelled by a multivariate Gaussian. Applying SMC allows for multiple hypotheses in the state-space, hence no Gaussian assumptions. A side effect is that the state-space can be searched in a more efficient manner, compared to standard iterative methods. Again, the reason being that multiple extrema do not pose a problem for the SMC methods, i.e., no Gaussian assumptions. SMC methods are implemented under different names, where especially the Condensation algorithm and the particle filter are popular. In chapter 8 a more detailed review of SMC methods can be found.

Year	First author	Initialisation	Tracking	Pose estimation	Recognition
2000	Barron			[4]	
2000	Buades			[8]	
2000	Chang		*	[9]	*
2000	Davis		[13]		
2000	Deutscher		*	[15]	
2000	Howe		*	[25]	
2000	Ormoneit	[43]	*	*	
2000	Ricquebourg		[53]		*
2000	Takahashi		[64]	*	
2000	Trivedi		[66]		
2000	Trivedi		*		[67]
2001	Ambrosio			[1]	
2001	Ambrosio			[2]	
2001	Barron			[5]	
2001	Choo			[11]	
2001	Delamarre		*	[14]	
2001	Elgammal	*	[16]		
2001	Grammalidis	*		[19]	
2001	Haritaoglu		*		[22]
2001	Herda	*	*	[23]	
2001	Hoshino		*	[24]	
2001	Huang		*	[26]	
2001	Intille				[28]
2001	Ioffe		*	[29]	
2001	Khan		[31]		
2001	Li			[32]	
2001	Mikić	*	*	[36]	
2001	Moeslund		*	[38]	

Year	First author	Initialisation	Tracking	Pose estimation	Recognition
2001	Moon		*	[41]	
2001	Ogaki		*	[42]	
2001	Pece	*	[45]		
2001	Plänkers		*	[48]	
2001	Rosales		*	[50]	
2001	Sangi		[56]		
2001	Sato		[57]		*
2001	Sidenbladh	*	*	[58]	
2001	Sminchisescu		*	[61]	
2001	Song			[62]	
2001	Song	[63]			
2001	Zhao		[70]		*
2002	Atsushi		[3]		
2002	Ben-Arie		*	*	[6]
2002	BenAbdelkader		*		[7]
2002	Cheng		*		[10]
2002	Davis		*	*	[12]
2002	Fua		*	[17]	
2002	Gleicher			[18]	
2002	Halvorsen		*	[20]	
2002	Hariadi		[21]		
2002	Huang		*	[27]	
2002	Jang		[30]	*	
2002	Li		*	[33]	
2002	Metaxas	[34]			
2002	Mikić	*	*	[35]	
2002	Mittal		[37]		
2002	Moeslund		*	[39]	
2002	Montemerlo		[40]		
2002	Ozer		[44]	*	*
2002	Pece		[46]		*
2002	Pers		[47]		
2002	Plänkers		*	[49]	
2002	Rao	*	*		[51]
2002	Ren		*	*	[52]
2002	Rittscher	*	*	*	[54]
2002	Roberts		*	[55]	
2002	Sidenbladh	*		[59]	
2002	Sminchisescu		*	[60]	
2002	Theobalt	*	*	[65]	
2002	Utsumi	*	[68]		
2002	Wang		[69]		
Σ	Total=70	3	20	37	10

Table A.1: Publications after September 2000 and their relation to the taxonomy in the survey.

Year	First author	Model type	Parts	Obj.	Ab. level	Dim.
2000	Barron [4]	Stick-Figure	22	Body	Joints	$2\frac{1}{2}$
2000	Deutscher [15]	Truncated Cones	17	Body	Edge/Silhouette	3
2000	Howe [25]	Planar Patches	14	Body	Motion	3
2001	Ambrosio [1]	Stick-Figure	16	Body	Joints	3
2001	Ambrosio [2]	Stick-Figure	16	Body	Joints	3
2001	Barron [5]	Stick-Figure	22	Body	Joints	$2\frac{1}{2}$
2001	Choo [11]	Stick-Figure	12	Body	Joints	3
2001	Delamarre [14]	Cones/Spheres	15	Body	Contours	3
2001	Grammalidis [19]	Ellipsoids	3	Arm	Depth	3
2001	Herda [23]	Stick-Figure	14	Body	Joints	3
2001	Huang [26]	Cylinders	10	Body	Silhouettes	3
2001	Li [32]	Truncated Cones	9	Body	Silhouettes	3
2001	Mikić [36]	Ellipsoids	10	Body	Depth	3
2001	Moeslund [38]	Cylinders	2	Arm	Silhouettes	3
2001	Moon [41]	Truncated Cones	15	Body	Edges	3
2001	Ogaki [42]	Stick-Figure	6	Arms	Silhouettes	3
2001	Plänklers [48]	Ellipsoids	~ 25	Body	Depth/Contour	3
2001	Sidenbladh [58]	Truncated Cones	4	Arms	Edges/Motion	3
2001	Sminchisescu [61]	Ellipsoids	15	Body	Edges/Motion	3
2001	Song [62]	Stick-Figure	15	Body	Joints/Motion	2
2002	Ben-Arie [6]	Ellipsoids	9	Body	Texture	2
2002	Fua [17]	Ellipsoids	~ 25	Arms	Depth/Contour	3
2002	Halvorsen [20]	Discs	28	Legs	Edges	3
2002	Huang [27]	Planar Patches	10	Body	Motion	2
2002	Mikić [35]	Ellipsoids	10	Body	Depth	3
2002	Plänklers [49]	Ellipsoids	~ 25	Arms	Depth/Contour	3
2002	Roberts [55]	Truncated Cones	14	Body	Texture	3
2002	Sidenbladh [59]	Truncated Cones	10	Body	Edges/Motion	3
2002	Sminchisescu [60]	Ellipsoids	15	Body	Edges/Silhouette	3
2002	Theobalt [65]	Cylinders (Points)	12	Body	Depth	3

Table A.2: Publications after September 2000 where a human model is used directly.

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Appendix B

Converting Rodrigues' Formula into Matrix Form

In this appendix a rotation-matrix representation of Rodrigues' formula is derived. The formula rotates a vector \vec{P} α degrees about a unit vector \vec{K} to a new vector \vec{P}' [1]. The rotation is given as

$$\vec{P}' = \vec{P} \cos(\alpha) + \sin(\alpha)(\vec{K} \times \vec{P}) + (1 - \cos(\alpha))(\vec{K} \cdot \vec{P})\vec{K} \quad (\text{B.1})$$

The task at hand is to rewrite equation B.1 to the form

$$\vec{P}' = \mathbf{R}_e(\alpha) \cdot \vec{P} \quad (\text{B.2})$$

Denoting $\vec{P} = [p_x, p_y, p_z]^T$ and $\vec{K} = [k_x, k_y, k_z]^T$ the second expression of equation B.1 can be rewritten as

$$\sin(\alpha)(\vec{K} \times \vec{P}) = \begin{bmatrix} \sin(\alpha)k_y p_z - \sin(\alpha)p_y k_z \\ -\sin(\alpha)k_x p_z + \sin(\alpha)p_x k_z \\ \sin(\alpha)k_x p_y - \sin(\alpha)p_x k_y \end{bmatrix} \quad (\text{B.3})$$

Combining equation B.3 and the first expression of equation B.1 yields

$$\begin{aligned} \vec{P}' \cos(\alpha) + \sin(\alpha)(\vec{K} \times \vec{P}) &= \begin{bmatrix} \sin(\alpha)k_y p_z - \sin(\alpha)p_y k_z + p_x \cos(\alpha) \\ -\sin(\alpha)k_x p_z + \sin(\alpha)p_x k_z + p_y \cos(\alpha) \\ \sin(\alpha)k_x p_y - \sin(\alpha)p_x k_y + p_z \cos(\alpha) \end{bmatrix} \Rightarrow \\ \vec{P}' \cos(\alpha) + \sin(\alpha)(\vec{K} \times \vec{P}) &= \begin{bmatrix} \cos(\alpha) & -\sin(\alpha)k_z & \sin(\alpha)k_y \\ \sin(\alpha)k_z & \cos(\alpha) & -\sin(\alpha)k_x \\ -\sin(\alpha)k_y & \sin(\alpha)k_x & \cos(\alpha) \end{bmatrix} \cdot \vec{P} \end{aligned} \quad (\text{B.4})$$

The last expression of equation B.1 is rewritten as

$$\begin{aligned}
(1 - \cos(\alpha))(\vec{K} \cdot \vec{P})\vec{K} &= (1 - \cos(\alpha))(k_x p_x + k_y p_y + k_z p_z)\vec{K} \Rightarrow \\
(1 - \cos(\alpha))(\vec{K} \cdot \vec{P})\vec{K} &= (1 - \cos(\alpha)) \begin{bmatrix} k_x^2 p_x + k_x k_y p_y + k_x k_z p_z \\ k_y k_x p_x + k_y^2 p_y + k_y k_z p_z \\ k_z k_x p_x + k_z k_y p_y + k_z^2 p_z \end{bmatrix} \Rightarrow \\
(1 - \cos(\alpha))(\vec{K} \cdot \vec{P})\vec{K} &= \\
\begin{bmatrix} (1 - \cos(\alpha))k_x^2 & (1 - \cos(\alpha))k_x k_y & (1 - \cos(\alpha))k_x k_z \\ (1 - \cos(\alpha))k_y k_x & (1 - \cos(\alpha))k_y^2 & (1 - \cos(\alpha))k_y k_z \\ (1 - \cos(\alpha))k_z k_x & (1 - \cos(\alpha))k_z k_y & (1 - \cos(\alpha))k_z^2 \end{bmatrix} \cdot \vec{P}
\end{aligned} \tag{B.5}$$

Adding equation B.4 and equation B.5, and writing each entry as $a \cdot \sin(\alpha) + b \cdot \cos(\alpha) + c$, where a, b and c are constants, yields

$$\begin{aligned}
\vec{P}' &= \mathbf{R}_e(\alpha) \cdot \vec{P} = \\
\begin{bmatrix} (1 - K_x^2) \cos(\alpha) + K_x^2 & -K_z \sin(\alpha) - K_x K_y \cos(\alpha) + K_x K_y \\ K_z \sin(\alpha) - K_x K_y \cos(\alpha) + K_x K_y & (1 - K_y^2) \cos(\alpha) + K_y^2 \\ -K_y \sin(\alpha) - K_z K_x \cos(\alpha) + K_z K_x & K_x \sin(\alpha) - K_y K_z \cos(\alpha) + K_y K_z \\ K_y \sin(\alpha) - K_z K_x \cos(\alpha) + K_z K_x & -K_x \sin(\alpha) - K_y K_z \cos(\alpha) + K_y K_z \\ (1 - K_z^2) \cos(\alpha) + K_z^2 & \end{bmatrix} \cdot \vec{P}
\end{aligned} \tag{B.6}$$

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Appendix C

Solution to the Transcendental Equation

Functions which are not algebraic are said to be transcendental. These contain one of the following expressions; logarithmic, exponential, hyperbolic, or trigonometric. In this chapter the trigonometric function of the form

$$a \cdot \sin(\alpha) + b \cdot \cos(\alpha) = c \quad (\text{C.1})$$

where a, b, and c are constants, is referred to as the transcendental equation and this appendix presents the solution to this equation.

The transcendental equation has two solutions within each interval of 360° , namely α_{min} and α_{max} .

To solve the equation the following substitution is used

$$u = \tan\left(\frac{\alpha}{2}\right) \quad \cos(\alpha) = \frac{1 - u^2}{1 + u^2} \quad \sin(\alpha) = \frac{2u}{1 + u^2} \quad (\text{C.2})$$

Inserting the substitution into equation C.1 and re-arranging yields

$$\begin{aligned} u^2(-b - c) + u(2a) + (b - c) &= 0 \Rightarrow \\ u &= \frac{a \pm \sqrt{a^2 + b^2 - c^2}}{b + c} \Rightarrow \\ \alpha_1 &= 2 \cdot \tan^{-1}\left(\frac{a + \sqrt{a^2 + b^2 - c^2}}{b + c}\right) \\ \alpha_2 &= 2 \cdot \tan^{-1}\left(\frac{a - \sqrt{a^2 + b^2 - c^2}}{b + c}\right) \end{aligned} \quad (\text{C.3})$$

This gives $\alpha_{min} = \min\{\alpha_1, \alpha_2\}$ and $\alpha_{max} = \max\{\alpha_1, \alpha_2\}$

When $c = 0$ a simpler expression can be derived

$$\begin{aligned} a \cdot \sin(\alpha) + b \cdot \cos(\alpha) &= 0 \Rightarrow \\ \frac{a \cdot \sin(\alpha)}{b \cdot \cos(\alpha)} + \frac{b \cdot \cos(\alpha)}{b \cdot \cos(\alpha)} &= 0 \Rightarrow \\ \frac{a}{b} \tan(\alpha) &= -1 \Rightarrow \\ \alpha &= \tan^{-1} \left(\frac{-b}{a} \right) \end{aligned} \tag{C.4}$$

The other solution is found as either $\alpha - 180^\circ$ or $\alpha + 180^\circ$